

mA 416

Lesson 29

Partition Numbers

Writing integer n as the sum of positive integers is called a partition of n

Ex $5 = 3 + 1 + 1$

$$5 = 2 + 2 + 1$$

$$5 = 5$$

Let P_n be the number of partitions of n

$$P_1 = 1 \quad P_2 = 2 \quad P_3 = 3 \quad P_4 = 5$$

For P_5 : $5, 4+1, 3+2, 3+1+1, 2+2+1,$
 $2+1+1+1, 1+1+1+1+1$

$$P_5 = 7$$

Notation: At times a partition is written as $k^{a_k} \dots z^{a_z} 1^{a_1}$ where a_i is the number of occurrence of i

in the sum:

$5 = 2+1+1+1 = 2^1 1^3$. This is just notation and we are not raising to powers

Let λ be a partition of $N = n_1 + \dots + n_k$

where $n_1 \geq n_2 \geq \dots \geq n_k$. This is how

we wrote the examples on the first page. We associate a

Ferrers diagram with the partition:

Put n_i dots in row i , with the diagram being left justified

$$\text{Ex } 8 = 3 + 2 + 2 + 1 \quad \begin{array}{ccccccc} & \bullet & \bullet & \bullet & & & \\ & \bullet & \bullet & & & & \\ & \bullet & \bullet & & & & \\ & & & & & & \end{array}$$

Theorem. Let $r \leq n$ be positive integers

let $P_n(r)$ be the number of partitions of n with the largest part (largest number in the partition) being r . let

$g_{n,r}(r)$ be the number of partitions of $n-r$ which has no part greater than r . Then $P_n(r) = g_{n,r}(r)$

We find a 1-1 correspondence between the two types of partitions mentioned in the theorem

Example of Theorem.

$$20 = \cancel{6+4+4+4+2}$$

$$\cancel{n=20} \quad r=6 \quad n-r=14$$

$$\cancel{P_n(r)} = \cancel{P_{20}(6)} = 5$$

$$n=10 \quad r=5$$

$$10 = 5 + 5$$

$$5+4+1$$

$$5+3+2$$

$$5+3+1+1$$

$$5+2+2+1$$

$$5+2+1+1+1$$

$$5+1+1+1+1$$

$$\cancel{P_{10}(5)} = 7$$

$$n-r = 10-5 = 5$$

$$5 = 5$$

$$= 4+1$$

$$= 3+2$$

$$= 3+1+1$$

$$= 2+2+1$$

$$= 2+1+1+1$$

$$= 1+1+1+1+1$$

$$\cancel{g_{10}(5)} = 7$$

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~~(Step)~~ Let λ be a partition of n with largest part r . Remove the part with r . We are left with a partition of $n-r$ with no part greater than r . To get the inverse, start with a partition of $n-r$ and no part longer than r . Add r to the partition to get a partition of the first type.

Ex $\begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \\ \bullet \end{array} \dots n$

$$\begin{array}{c} \bullet \bullet \\ \bullet \bullet \\ \bullet \bullet \end{array} \rightarrow \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \\ \bullet \bullet \end{array}$$

$$\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \dots$$

$$n=12 \quad r=5$$

$$n-r=7 \quad \text{all parts} \leq r$$

Conversely

$$\begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \\ \bullet \end{array} \text{ becomes } \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \\ \bullet \end{array} \text{ giving back}$$

the original. This completes the proof.

Note in partition notation

$$12 = 5+3+3+1 \rightarrow n-r=7 = 3+3+1$$

$$r=5$$

Converse add r

$$7 = 3+3+1$$

$$12 = 5+3+3+1$$

to get inverse map

5 Given a partition λ , its conjugate λ^* is the partition obtained by interchanging rows and columns with each other.

$$\lambda \begin{array}{c} \vdots \vdots \vdots \\ \vdots \vdots \vdots \\ \vdots \vdots \vdots \\ \vdots \vdots \vdots \end{array} \rightarrow \lambda^* \begin{array}{c} \vdots \vdots \vdots \\ \vdots \vdots \vdots \\ \vdots \vdots \vdots \\ \vdots \vdots \vdots \end{array}$$

The largest part in λ is the number of parts in λ^* . It can be seen by reflecting through the diagonal from the $(1,1)$ position to the last position on the diagonal.



Reflect through line

A partition is self conjugate if

$$\lambda = \lambda^*$$

After reflecting, we get the same diagram.

Ex

$$\lambda \begin{array}{c} \vdots \vdots \vdots \\ \vdots \vdots \vdots \\ \vdots \vdots \vdots \\ \vdots \vdots \vdots \end{array} \quad \lambda^* \begin{array}{c} \vdots \vdots \vdots \\ \vdots \vdots \vdots \\ \vdots \vdots \vdots \\ \vdots \vdots \vdots \end{array} \quad \lambda = \lambda^*$$

6 Theorem. Let n be a positive integer
 P_n^S = number of self conjugate partitions
 of n and P_n^T be the number
 of partitions with n distinct
 odd parts (Each row has an odd
 number of elements) Then
 $P_n^S = P_n^T$

We set up a 1-1 correspondence between
 the two sets in Theorem, call
 them S and O . Let $\lambda \in S$
 It is self conjugate. The total
 number of elements in row 1 +
 total number of elements in column 1
 is odd. ~~List~~ together as the first
 row in $\mu \in O$. Delete this row and
 column and repeat process. The new
 row in μ is odd and has fewer elements
 than the first row. Continue down to
 the last position on the diagonal of λ
 Since $\lambda = \lambda^*$, all elements are used

$$\begin{matrix} & \cdots & \cdots & \cdots & \cdots & 9 \\ & \cdots & \cdots & \cdots & \cdots & 5 \\ & \cdots & \cdots & \cdots & \cdots & \end{matrix}$$

Distinct odd parts, 9 and 5

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Going back, the first row is bent at the mid point (there are an odd number of dots in each row), giving the first row and column of the self conjugate partition. Do this with the second row until all dots are used.

Theorem (Euler) Let n be a positive integer. Let P_n^o be the number of partitions of n into odd parts and P_n^d the number of partitions into distinct parts. Then

$$P_n^o = P_n^d$$

Proof Again we set a 1-1 correspondence. Suppose λ has all odd parts. If the parts are distinct, done. If there are 2 copies of the same part, combine them into one part. Continue this until all parts are distinct. This gives a partition of the second type in the theorem.

To go back, consider a partition into distinct parts. If all parts are odd, done. If not, there is an even part. Split it into 2 equal parts. Repeat until all parts are odd. Hence we have a partition of the first type.

$$\text{Ex} \quad 20 = 7 + 5 + 5 + 3$$

$$\begin{aligned} \rightarrow &= 7 + 10 + 3 \\ &= 10 + 7 + 3 \end{aligned}$$

$$\begin{aligned} \text{Conversely } 20 &= 10 + 7 + 3 \\ &= 5 + 5 + 7 + 3 \\ &= 7 + 5 + 5 + 3 \end{aligned}$$

Setting back the original

See page 295 for another example

We construct a generating function for the sequence of partition numbers

It comes from an infinite product

$$\text{Theorem } \sum_{n=0}^{\infty} P_n x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

The right hand side has terms

$$\frac{1}{1-x^k} = 1 + x^{k \cdot 1} + x^{k \cdot 2} + x^{k \cdot 3} + x^{k \cdot 4} + \dots$$

So the right hand side is

$$(1 + x^1 + x^{2 \cdot 1} + x^{3 \cdot 1} + \dots) (1 + x^{2 \cdot 1} + x^{2 \cdot 2} + x^{3 \cdot 2} + \dots) \cdots (1 + x^{k_1} + x^{k_2} + \dots)$$

Looking for P_n we take one number from each sum and multiply the ones chosen. All but a finite number are 1. The sum of the exponents is n . This hold for each pass through the infinite series.

and the sum is the answer for P_n

$$n=4$$

$(1+x+x^2+x^3)(1+x^2+x^4)(1+x^3)(1+x^4)$ are the terms not equal to 1. Collecting

~~$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8$~~ or

$$(1+x^{1.1}+x^{2.1}+x^{3.1}) (1+x^{1.2}+x^{2.2}+x^{3.2})$$

$$(1+x^{1.3}+x^{2.3}+x^{3.3}) (1+x^{1.4}+x^{2.4}+ \dots)$$

To get the coefficient of x^4 :

$$x^{1.1} \rightarrow 1+1+1$$

$$x^{2.2} \rightarrow 2+2$$

$$x^{2.1}x^{1.2} \rightarrow 1+1+2$$

$$x^{1.1}+x^{1.3} \rightarrow 1+3$$

$$x^{1.4} \rightarrow 4$$

$$\text{So } P_{st} = 5$$

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Let P_n be the set of partitions of n . We will partial order P_n

$$\text{Let } \lambda \Rightarrow n = n_1 + n_2 + \dots + n_k \quad n_1 \geq n_2 \geq \dots \geq n_k$$

$$\mu : n = m_1 + m_2 + \dots + m_l \quad m_1 \geq m_2 \geq \dots \geq m_l$$

If the partitions are of different length, put some 0's at the end of the shortest one so that they now have the same length

λ is majorized by μ if the partial sums of λ are at most equal to the partial sums of μ . i.e

$$n_1 + n_s \leq m_1 + \dots + m_s \quad \text{for } s = 1, 2, \dots, k$$

Ex Partitions of 9

$$\lambda \quad 9 = 5 + 1 + 1 + 1 + 1$$

$$\mu \quad 9 = 4 + 2 + 2 + 1 + 0$$

$$\nu \quad 9 = 4 + 4 + 1 + 0 + 0$$

$\mu \leq \nu$ but λ and μ are not comparable nor are λ and ν

This is a partial ordering as it is easily seen to be reflexive, antisymmetric and transitive

Def. Let $\lambda: n = n_1 + n_2 + \dots + n_k$ $\mu: m = m_1 + m_2 + \dots + m_k$

$$n_1 \geq n_2 \geq \dots \geq n_k \quad m_1 \geq m_2 \geq \dots \geq m_k$$

λ precedes μ in the lexicographic order provided there is an integer t such that $n_j = m_j$ for all $j < t$ and $n_t < m_t$.

Ex $12 = 4+3+2+2+1$ precedes
 $12 = 4+3+3+1+1$ since

$$4=4, 3=3 \text{ but } 2<3$$

This order is reflexive, anti-symmetric and transitive, hence λT is a Partial order on P_n

Theorem. Lexicographic order is a linear extension of majorization on P_n

Proof Lexicographic ordering is clearly a Total order as each pair of partitions are comparable.

Suppose λ is majorized by μ . Choose first t such that $n_j = m_j$ for $j < t$ and $n_t + m_t$. (λ and μ as in Def at top of page)

Since $n_{i+} + n_{t-} + n_t \leq m_{i+} + m_{t-} + m_t$

and $n_i = m_i$, $i < t$

$\rightarrow n_t < m_t$ and \rightarrow proceeds

μ in the lexicographic order

Problems Ch. 8 21, 22, 23, 24

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