

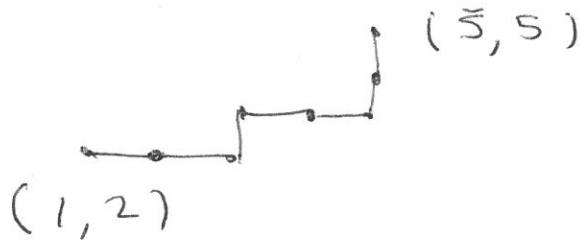
Lesson 30

Mon 4/6

# Lattices

The set of all points in the plane with integer coordinates is called a lattice.

Given points  $(r, s)$ ,  $(p, q)$  with  $p \geq r$  and  $q \geq s$ , a lattice path from  $(p, q)$  to  $(r, s)$  is a sequence of moves that are horizontal of length +1 or vertical of length +1



How many lattice paths are there from  $(r, s)$  to  $(p, q)$ ? In the example the lattice path can be represented by

where  $H = (1, 0)$  and  $V = (0, 1)$ . Any lattice path from  $(r, s)$  to  $(p, q)$  will have 4 H's and 3 V's in some order. We are counting the number of multiset permutations  $\binom{4+3}{4, 3} = \frac{7!}{4! 3!}$

Generally, letting  $p-r=m$ ,  $q-s=n$

The number of lattice path is

$$\binom{m+n}{m \ n} = \frac{(m+n)!}{m! \ n!}$$

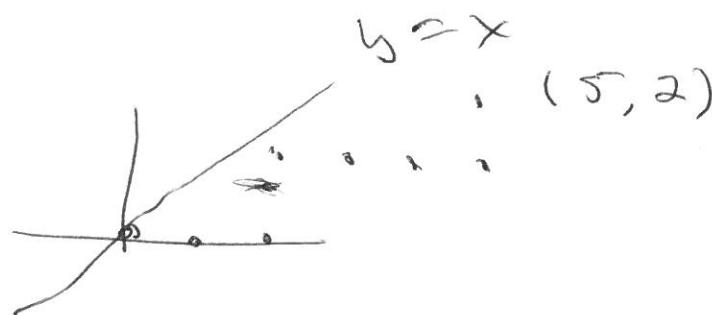
Next, consider the number of lattice paths from  $(r,s)$  to  $(p,q)$  that are below  $y=x$ . This forces  $r \geq s$  and  $p \geq q$  otherwise one of the points is above  $y=x$ . The number from  $(0,0)$  to  $(n,n)$  was shown to be  $C_n = \frac{1}{n+1} \binom{2n}{n}$ ,  
C<sub>n</sub> the n'th Catalan number

Consider the number of lattice paths from  $(0,0)$  to  $(p,q)$ ,  $p \geq q$ , that are below  $y=x$  (called subdiagonal lattice paths)

Theorem. Let  $p$  and  $q$  be positive integers,  $p \geq q$ . The number of subdiagonal paths from  $(0,0)$  to  $(p,q)$  is

$$\frac{p-q+1}{p+1} \binom{p+q}{q}$$

An example



Proof. As in the earlier proof we count the number of lattice paths that cross  $y=x$  and subtract that from the total number of paths from  $(0,0)$  to  $(p,q)$ . The last number is  $\binom{p+q}{q}$

Let  $d(p,q)$  be the number that cross  $y=x$ . This is the

+ same as the number of paths  
from  $A: (0, -1)$  to  $B(P, q-1)$

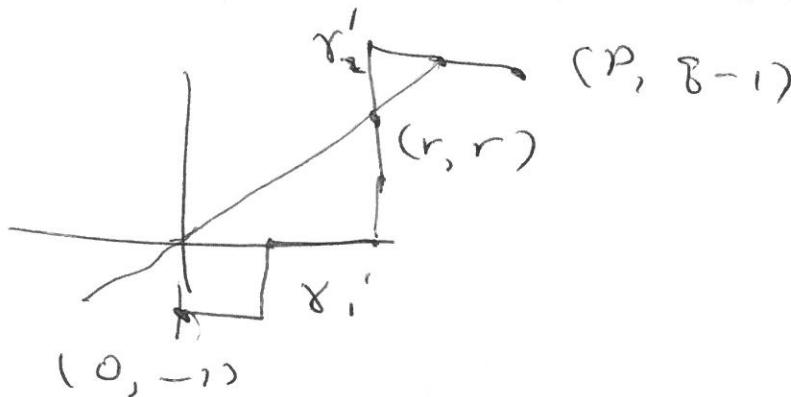
that touch or cross  $y=x$ .

Let  $\gamma'$  be a path from  $(0, -1)$   
to  $(P, q-1)$  that touches  $y=x$

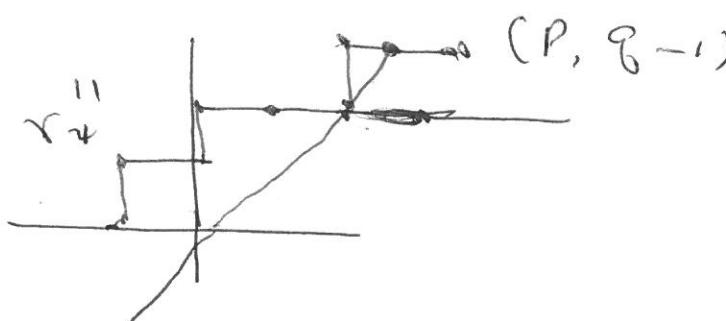
for the first time at  $(r, r)$

let  $\gamma'_1$  be the path up to  $(r, r)$

and  $\gamma'_2$  be the rest of the path



Reflect  $\gamma'_1$  through  $y=x$  to get  $\gamma''_1$



The new path goes from  $(-1, 0)$   
thru  $(r, r)$  to  $(P, q-1)$

5 It must hit  $y=x$ . Since  $(-1, 0)$  and  $(P, q-1)$  are on opposite sides we reverse the process by reflecting the first part of the path back. So the number of paths we are looking for = the number of all paths from  $(-1, 0)$  to  $(P, q-1) = \binom{P+q-1}{q-1} = \binom{P+q}{q-1}$

$$\begin{aligned} \text{So the number of sub diagonal paths from } (0, -1) \text{ to } (P, q-1) &= \\ \binom{P+q}{q} - \binom{P+q}{q-1} &= \frac{(P+q)!}{q! P!} - \frac{(P+q)!}{(q-1)! (P+1)!} \\ &= \frac{P-q+1}{P+1} \binom{P+q}{q} \end{aligned}$$

Now also allow diagonal steps  $(1, 1)$   
 Such a path is called a  
 HVD lattice path. Count  
 $K(p, q)$ , the number of HVD  
 paths from  $(0, 0)$  to  $(p, q)$   
 and  $K(p, q, r)$ , the number with  
 $r$  diagonal steps. When  $r=0$ ,  
 $K(p, q, 0) = K(p, q) = \binom{p+q}{q}$  from  
 before.

Theorem. Let  $r = \min(p, q)$ . Then

$$K(p, q, r) = \binom{p+q-r}{p-r, q-r, r} = \frac{(p+q-r)!}{(p-r)!(q-r)!r!}$$

and

$$K(p, q) = \sum_{r=0}^{\min(p, q)} \frac{(p+q-r)!}{(p-r)!(q-r)!r!}$$

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Now we also allow diagonal steps  $(1,1)$ . Such a path is called a HV D-lattice path

We count  $\mathbb{K}(p,q)$  be the number of HV D lattice paths from  $(0,0)$  to  $(p,q)$  and  $\mathbb{K}(p,q;r)$  be the number with  $r$  Diagonal steps

When  $r=0$   $\mathbb{K}(p,q,0) = \mathbb{K}(p,q)$  the number of lattice paths that we have computed:  $\mathbb{K}(p,q,0) = \binom{p+q}{p}$

Thm. Let  $r = \min(p,q)$  Then

$$\mathbb{K}(p,q,r) = \binom{p+q-r}{p-r \quad q-r \quad r} = \frac{(p+q-r)!}{(p-r)!(q-r)!r!}$$

and

$$\mathbb{K}(p,q) = \sum_{r=0}^{\min(p,q)} \frac{(p+q-r)!}{(p-r)!(q-r)!r!}$$

$$r=0$$

Proof. Given the number of diagonal steps to be  $r$ , there are  $p-r$  horizontal steps and  $q-r$  vertical steps. We are looking for the number of multiset permutations

of  $\{(p-r)H, (q-r)V, rD\}$ . That is

$$\binom{(p+r)+(q-r)+r}{(p-r) \quad q-r \quad r} = \frac{(p+q-r)!}{(p-r)!(q-r)!r!} = K(p, q, r)$$

To set up paths  $r$  moves from  $0$  to  $\min(p, q)$ , giving

$$\begin{cases} \min(p, q) \\ K(p, q, r) \end{cases}$$

$$r=0$$

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Ex How many paths with diagonal?  
 Paths possible are there from  
 $(0,0)$  to  $(3,5)$

$$\begin{aligned}
 & K(3,5,0) + K(3,5,1) + K(3,5,2) + K(3,5,3) \\
 = & \frac{8!}{3!5!} + \frac{7!}{2!4!} + \frac{6!}{1!3!2!} + \frac{5!}{0!2!3!}
 \end{aligned}$$

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Now we consider subdiagonal paths when diagonal moves are allowed

Let  $R(p, q)$ ,  $p \geq q$  be the number of subdiagonal moves from  $(0,0)$  to  $(p,q)$  where diagonal moves are allowed and  $R(p, q, r)$  be the number when exactly  $r$  diagonal moves are allowed

$$\text{Evidently } R(p, q) = \sum_{r=0}^q R(p, q, r)$$

Then

$$\text{Thm } R(p, q, r) = \frac{p-q+r}{p-r+1} \frac{(p+q+r)!}{r!(p-r)!(q-r)!}$$

$$\text{and } R(p, q) = \sum_{r=0}^q \frac{p-q+r}{p-r+1} \frac{(p+q+r)!}{r!(p-r)!(q-r)!}$$

Proof A subdiagonal lattice path  $\gamma$  from  $(0,0)$  to  $(p,q)$  with  $r$  diagonal steps becomes a subdiagonal regular path from  $(0,0)$  to  $(p-r, q-r)$  (No diagonal steps)

Conversely given a regular lattice path from  $(0,0)$  to  $(p-r, q-r)$  becomes a lattice path with diagonal moves from  $(0,0)$  to  $(p,q)$  if  $r$  diagonal moves are inserted in any of the  $p+q-2r+1$  places in the rectangular path

The number of insertions possibilities is the number of solutions to  $x_1 + x_2 + \dots + x_{p+q-2r+1} = r$

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As we know, this is

$$\binom{P+Q-2r+1+r-1}{r} = \binom{P+Q-r}{r}$$

Hence, each subdiagonal lattice path with just rectangular moves from  $(0,0)$  to  $(P-Q, Q-r)$  there are  $\binom{P+Q-r}{r}$  paths from  $(0,0)$  to  $(P, Q)$  with diagonal moves allowed. Thus

$$R(P, Q, r) = \binom{P+Q-r}{r} R(P-r, Q-r, 0) = \\ \binom{P+Q-r}{r} \frac{P-Q+1}{P-r+1} \binom{P+Q-2r}{Q-r} = \\ \frac{P-Q+1}{P-r+1} \binom{P+Q-r}{r \ P-r \ Q-r}$$

If  $P=Q=n$ , these paths are called Schröder paths. The large Schröder number,  $R_n$  is the number of Schröder paths from  $(0,0)$  to  $(n,n)$ . Thus

$$R_n = \sum_{r=0}^n \frac{1}{n-r+1} \frac{(2n-r)!}{r!((n-r)!)^2}$$

Ex. Partial Ordered Sets for  $n=6$

Lex

$$\begin{array}{c}
 6 \\
 | \\
 5+1 \\
 | \\
 4+2 \\
 | \\
 4+1+1 \\
 | \\
 3+3 \\
 | \\
 3+2+1 \\
 | \\
 3+1+1+1 \\
 | \\
 2+2+2 \\
 | \\
 2+2+1+1 \\
 | \\
 2+1+1+1+1 \\
 | \\
 1+1+1+1+1+1
 \end{array}$$

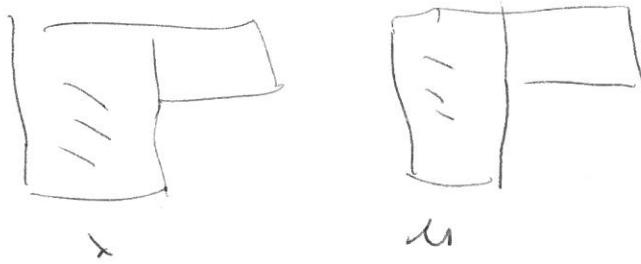
Majorization

$$\begin{array}{c}
 6 \\
 | \\
 5+1 \\
 | \\
 4+2 \\
 | \\
 4+1+1 \\
 | \\
 3+2+1 \\
 | \\
 2+2+2 \\
 | \\
 3+1+1+1 \\
 | \\
 2+2+1+1 \\
 | \\
 2+1+1+1+1 \\
 | \\
 1+1+1+1+1+1
 \end{array}$$

Then, let  $\lambda$  and  $\mu$  be partitions of  $n$ . If  $\mu$  majorizes  $\lambda$ , then  $\lambda^*$  majorizes  $\mu$ .

Proof. Suppose not. Then there are columns 1, 2, ...,  $s$  in  $\lambda$  and in  $\mu$  such that the sum of the first is less than the sum of the second

Graphically



Removing the first  $s$  columns from each, what is left of  $\lambda$ ? What is left of  $\mu$ . Call these  $\bar{\lambda}$  and  $\bar{\mu}$ . Let  $t$  be the last non zero row in  $\bar{\lambda}$ . The sum of these rows  $>$  sum of same rows in  $\bar{\mu}$ .

Contradiction.