

Ma 416

Lesson 31

Graphs

A graph G consists of a set V of vertices and a set E of pairs of elements from V , called edges.

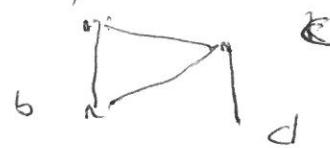
If the pairs are not ordered, then the graph is undirected and these are the graphs we will consider. There are also directed graphs where the pairs are ordered.

$$\text{Ex } G = (V, E) \quad V = \{a, b, c, d\}$$

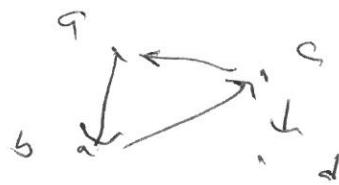
$$E = \{(a, b), (b, c), (c, a), (c, d)\}$$

If the edges are not ordered

Then G is



If they are ordered



We will specify if the graph is directed

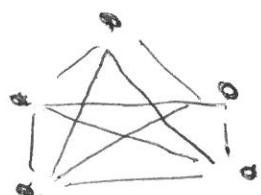
We will deal with non directed graphs. The graph is simple if there are no multiple edges between any pair of vertices. The order of a graph is the number of vertices.

If there are multiple edges between a pair of vertices, then the graph is called a multigraph.

The multiplicity of an edge is the number of times it joins vertices a and b . If edges (a,a) are allowed, the edge is called a loop.

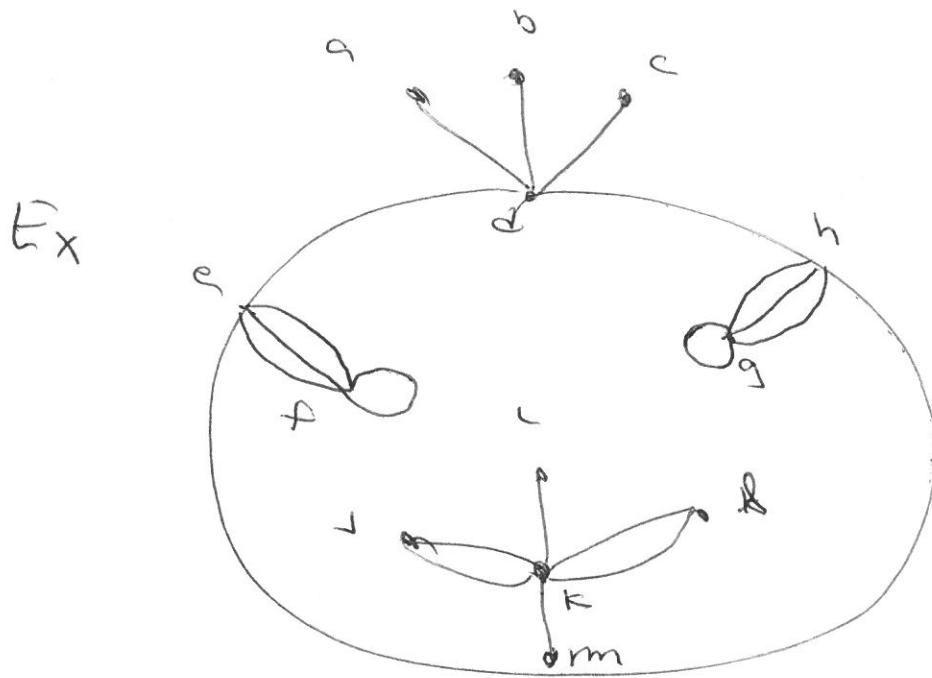
A graph is called complete if between each pair of vertices there is an edge.

Ex

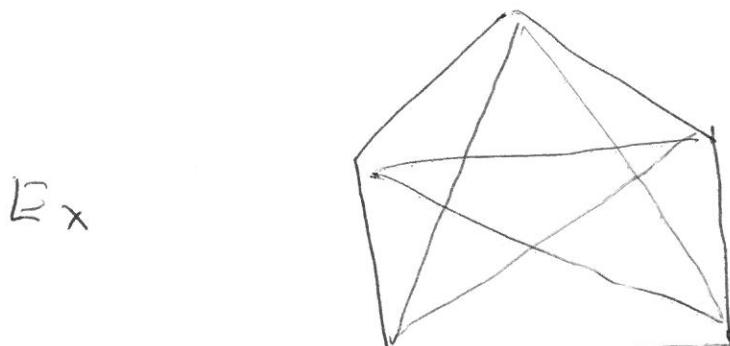


A complete graph of order n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges and is denoted by K_n .

On page 398 of the text, the Graph Buster graph appears
It has order 13 and 21 edges



It is clearly a multigraph with loops.



K_5

* The degree of vertex x in a graph is the number of edges that have x as an end vertex.
(this is described as the edge being incident with the vertex)

When the edge is a loop it contributes 2 to the degree.



$$\deg(a) = 3 \quad \deg b = 1 \quad \deg c = 1$$

To graph G there is the degree sequence which is a listing of the degrees of the vertices in nonincreasing order. In the above example $\{d_1, d_2, d_3\} = \{3, 1, 1\}$.
The degree sequence of K_n is $\{n-1, \dots, n-1\}$.

5 Theorem. Let G be a graph. The sum of the degrees of all the vertices is an even number.

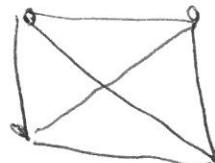
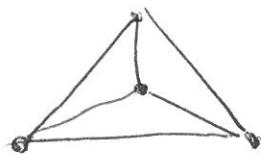
The number of vertices of odd degree is even.

Proof. Each edge contributes 2 to the sum of the degrees. Hence the first part of the Theorem holds. The second part follows from the first.

Two graphs are isomorphic,
 $G = (V, E)$ $G' = (V', E')$ if there is a 1-1 mapping $V \rightarrow V'$ such that for each pair $x, y \in V$, with images $x', y' \in V'$, the number of edges joining x to y equals the number of edges joining x' to y' . The mapping is called an isomorphism.

6 If G and G' are simple graphs (no multiple edges or loops), then G and G' are isomorphic if there is a mapping $V \rightarrow V'$ such that vertices are adjacent in V if and only if they are adjacent in V'

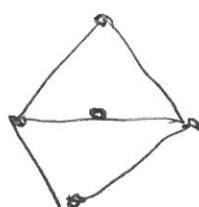
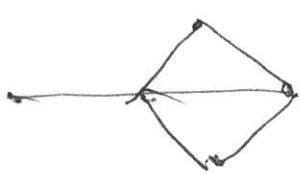
Ex.



These graphs of order 4 are isomorphic. They are simple graphs and all pairs of vertices are adjacent in each of them.

Ex. Graphs with the same number of vertices and edges,

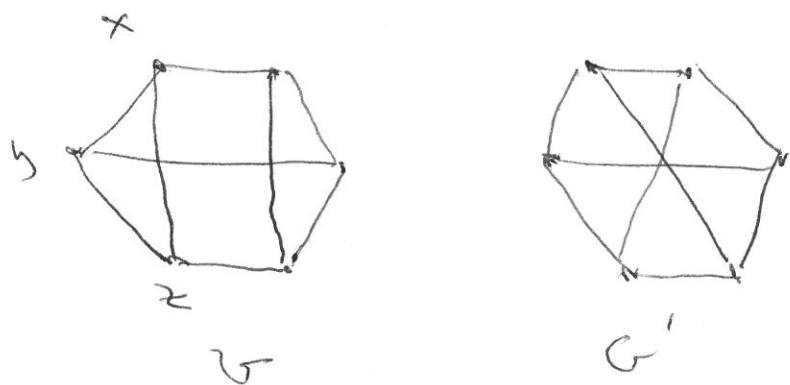
but are not isomorphic



The first graph has a vertex of degree 1, but the second does not. There is no way to define

a mapping which is an isomorphism because the vertex of degree 1 must go to a vertex of degree 1, impossible.

Example Isomorphic graphs have the same degree sequence, but the converse does not hold



In G , x, y, z are all adjacent. If $\Theta: G \rightarrow G'$ is an isomorphism, then $\Theta(x), \Theta(y), \Theta(z)$ would be adjacent. There are no 3 such points in G' . However the order of both graphs are 6 with degree sequence

$$(3, 3, 3, 2, 2, 3)$$

Def Let $x_0, \dots, x_m \in V$ and

$(x_0, x_1), (x_1, x_2) \dots (x_{m-1}, x_m) \in E$.

This situation is called a walk
of length, m joining x_0 and x_m

Denote it by

$$x_0 - x_1 - \dots - x_m$$

If $x_0 = x_m$, the walk is closed
otherwise it is open

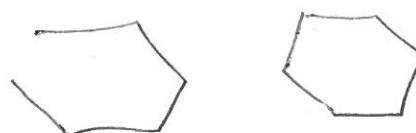
If the walk has distinct
edges it is called a trail

If the walk has distinct
vertices (except, perhaps, $x_0 = x_m$)
it is called a path.

A closed path is a cycle



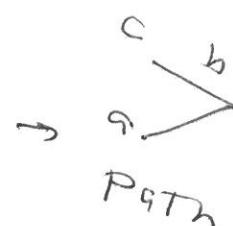
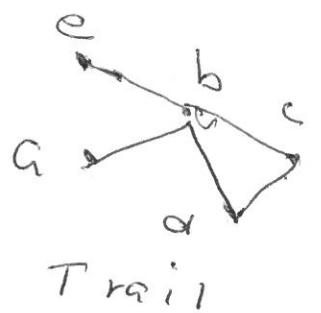
Trail



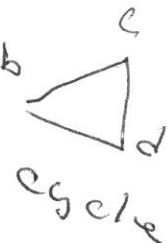
Path



Cycle



Path



Cycle

Examples in Graph Tester

a - d - b - d - c - d - h - g - h - m - k - i

watts walk not a Trail

a - d - e - f - e - m - k - l - k - i

Trail not a path

a - d - e - m - k - i

path

d - e - f - e - m - h - d

closed Trail, not cycle

d - e - m - h - d

cycle

e - f - e

cycle

f - f

cycle

Def A graph is connected if for each $x, y \in V$, there is a walk joining x and y . $d(x, y)$ is the shortest length of a walk from x to y . $d(x, y)$ is the distance from x to y . Define $d(x, x) = 0$. A walk joining x to y of shortest distance is a path.

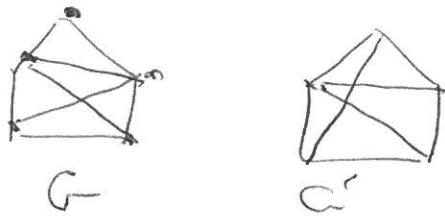
Def Let $G = (V, E)$ be a graph. A subset V' of V and a submultiset E' of E is called a subgraph $G' = (V', E')$ of G . If E' has all edges that join the vertices in V' then G' is the induced subgraph of G . If V' is all of V , then G' is called spanning.

Theorem. Let G and G' be graphs

Suppose G and G' are isomorphic
Then

1. If G is simple, Then G' is simple
(simple = at most one edge between vertices)
2. If G is connected, Then G' is connected
3. If G has a cycle of length k , then so does G'
4. If G has an induced subgraph that is a K_n , so does G'

Ex.



G has a K_4 subgraph

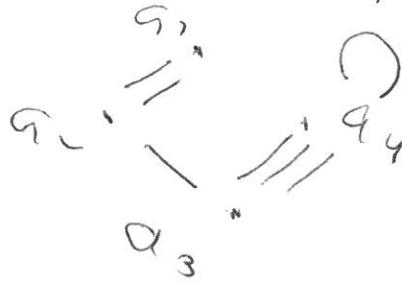
G' does not

G and G' are not isomorphic

12 Let G be graph with vertices v_1, \dots, v_n . Let a_{ij} be the number of edges between v_i and v_j .
 $A = (a_{ij})$ is called the adjacency matrix. $a_{ii} =$ number of loops at v_i and when the graphs are not directed, $a_{ij} = a_{ji}$.

If G and G' have the same adjacency matrix, then G and G' are isomorphic. Let $\Theta: V_i \rightarrow V'_i$ where A and A' come from ordering v_1, \dots, v_n and v'_1, \dots, v'_n .

Ex $V = \{q_1, q_2, q_3, q_4\}$



$$A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 1 \end{pmatrix}$$

13

Recall that any partition of n gives a Ferrer's diagram.

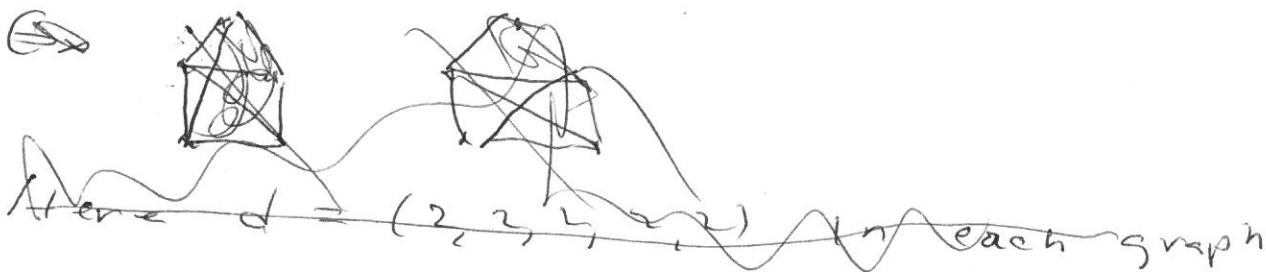
$$\text{Ex } n = (5, 3, 2, 2, 1)$$



The number of rows is called the number of parts in the partition.

Then a diagram also gives a partition of n where n is the number of dots in the diagram.

As we have seen, non-isomorphic graphs can have the same number of vertices and the same degree sequence.



Ex



For example $d = (3, 3, 3, 3, 3, 3)$ in both examples and the order is 6 but they are not isomorphic.

So a graph has a degree sequence which gives a Ferrer's diagram. But a Ferrer's diagram may not have a graph (even though it has a partition of n)

Ex.  $d = (3, 2, 2)$. If this

has a graph, it has $\frac{3+2+2}{2}$ edges

which is impossible. For a diagram or partition to be the degree sequence of a graph, the number of dots must be even (the ~~no~~ sum of the numbers in d must be even). In fact, a partition is called graphic if there is a graph whose degree sequence is the partition.

The trace of λ (or associated diagram γ) is the number of positions on the diagonal of Γ

Theorem (Ruch-Gutman) Suppose λ

is a partition of an even number n . Then $\lambda = (\lambda_1, \dots, \lambda_n)$ is graphic if and only if $\sum_{l=1}^r \lambda_l \leq \sum_{l=1}^r (\lambda_l^* - 1)$

where $1 \leq r \leq t_n(\lambda)$.

Proof Suppose λ is graphic with conjugate partition λ^* . Number the vertices in the graph such that λ_l is the number of edges at vertex i . Let Y be the associated Young diagram. In each row l of Y place in increasing order the numbers of the vertices with edges to vertex i . The first column in Y contains all the 1's (for the first vertex) as well as a number greater than 1 in the $(1, 1)$ position. Hence $\lambda_1^* \geq \lambda_1 + 1$.



$$\lambda = (4, 4, 3, 3, 2, 2)$$

$$\begin{matrix} Y(\lambda) & 3 & 4 & 5 & 6 \\ & 3 & 4 & 5 & 6 \\ & 1 & 2 & 4 \\ & 1 & 2 & 3 \\ & 1 & 2 \\ & 1 & 2 \end{matrix}$$

The number of 1's (≤ 11 in first col) $\leq \lambda_1^{* - 1}$

$$\begin{matrix} \lambda_1 & 11 \\ \lambda_2 & 11 \\ \lambda_3 & 11 \end{matrix} \quad \text{The number of } 1's + 2's \quad \text{(all in cols 1 and 2)} \\ \lambda_1^{* - 1} + \lambda_2^{* - 1}$$

Continue until we get last diagonal entry

$$\lambda_1 + \lambda_2 = \# 1's + 2's + 3's \leq \lambda_1^{* - 1} + (\lambda_2^{* - 1}) + (\lambda_3^{* - 1})$$

17 We do not prove the converse
but do prove a special case

Def Let λ be a partition of n
where n is even. λ is
called a Threshold partition
if $\lambda_i = \lambda_1^* - 1$ for $i=1, \dots, \text{tr}(\lambda)$

Let λ be a threshold partition

$$\begin{matrix} G_x & \begin{matrix} \bullet & \bullet & \cdots & \bullet \\ \vdots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & & \vdots \\ \vdots & \ddots & & \end{matrix} \end{matrix}$$

Remove the first row and
column. This removes λ_1 and
subtracts 1 from all other λ_i

$$\begin{matrix} \bullet & \bullet & \cdots & \bullet \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & \ddots \end{matrix} \quad \text{Call it } \lambda'$$

This is another threshold partition
diagram

Repeat $\vdots \vdots \lambda'$

For each \circ at the bottom of

By induction λ' has a graph with degree sequence λ' .

$$\text{Ex } \lambda = (5, 4, 4, 3, 3, 1)$$

$$\begin{matrix} Y & \begin{matrix} 5 & 4 & 4 & 3 & 3 & 1 \\ 5 & 4 & 4 & 3 & 3 & 1 \\ 4 & 3 & 3 & 2 & 2 & 1 \\ 3 & 2 & 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{matrix} \end{matrix}$$

Remove first row
and column

$$\begin{matrix} 5 & 4 & 4 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \end{matrix}$$

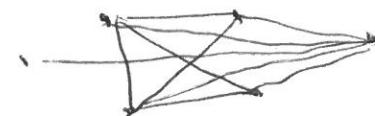
This is threshold diagram. We construct it's graph



To go back, there is an isolated vertex



Now add vertex
and connect to all



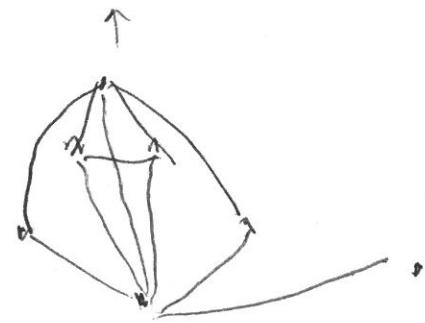
Deg sequence
(5, 4, 4, 3, 3, 1)

We can also construct the graph not using induction, but reduce graph to trace = 1

$$\lambda = (6, 5, 3, 3, 2, 2, 1)$$

$\begin{matrix} a & a & a & a & a & a \\ a & a & a & a & a & a \\ a & a & a & a & a & a \\ a & a & a & a & a & a \\ a & a & a & a & a & a \\ a & a & a & a & a & a \end{matrix}$
 Remove
row + col

$$\lambda = (6, 5, 3, 3, 2, 2, 1)$$



$\begin{matrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{matrix}$
 Remove
row + col

Add vertex connect to all

One isolated



2 isolated

Add vertex, connect to all



Co to top diagram