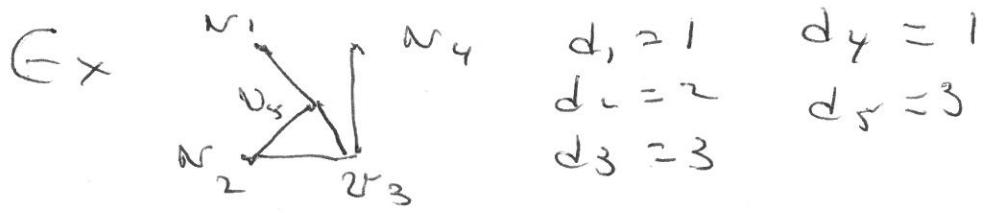


MA 416

Lesson 32

Threshold Graphs

Let's review what we know about the relations between graphs, partitions of  $n$  and Ferrer's diagrams. Let  $G$  be a graph with vertices  $N_1, \dots, N_n$ . Let  $d_j$  be the number of edges incident with  $N_j$  and let  $d = (\lambda_1, \lambda_2, \dots, \lambda_n)$  be the  $d_j$  listed in non decreasing order  $\sum \lambda_i = 2k$  where  $k$  is the number of edges.  $\lambda_1 + \dots + \lambda_n = 2k$  is a partition of  $2k$ . Any partition of any positive integer  $s$  can be displayed in a Ferrer's diagram of  $n$  left justified rows where the  $i$  row has length  $\lambda_i$ .



$d = (3, 3, 2, 1, 1)$  diagram



2) Each diagram gives a partition by listing the number of dots in each row. In the example that would be  $(3, 3, 2, 1, 1)$ . The question is can we go from a Ferrer's diagram to a graph. If the sum of the dots is odd, the answer is never. Even if the sum is even, it is not always possible. Indeed we have the following theorem.

(Remember  $\lambda_i$  = number of rows in the diagram and  $\lambda_i^*$  is the number of columns in the diagram. Also the trace is the number of elements on the diagonal of the diagram)  
(A partition is graphic if it is the degree sequence of some graph)

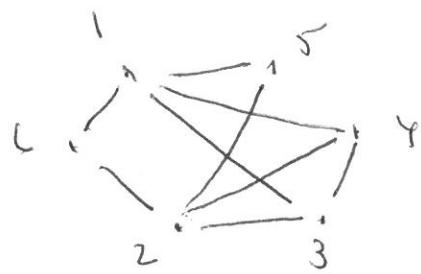
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Theorem (Ruch-Gutman) Suppose  $\lambda$  is a partition of an even number  $n$ . Then  $\lambda = (\lambda_1, \dots, \lambda_n)$  is graphic if and only if  $\sum_{i=1}^r \lambda_i = \sum_{i=1}^r (\lambda_i^* - 1)$  where  $1 \leq r \leq \text{tr}(\lambda)$ .

Proof Suppose  $\lambda$  is graphic with conjugate partition  $\lambda^*$ . Number the vertices in the graph such that  $\lambda_i$  is the number of edges at vertex  $i$ . Let  $\gamma$  be the associated Young diagram. In each row  $i$  of  $\gamma$  place in increasing order the numbers of the vertices with edges to vertex  $i$ . The first column in  $\gamma$  contains all the 1's (for the first vertex) as well as a number greater than 1 in the  $(1, 1)$  position. Hence  $\lambda_i^* \geq \lambda_{i+1}$

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Example



$$\lambda = (4, 4, 3, 3, 2, 2)$$

$$\begin{matrix} Y(\lambda) & 3 & 4 & 5 & 6 \\ & 3 & 4 & 5 & 6 \\ & 1 & 2 & 4 \\ & 1 & 2 & 3 \\ & 1 & 2 \\ & 1 & 2 \end{matrix}$$

The number of 1's ( $\leq 11$  in first col)  $\leq \lambda_1^{*+1}$

The number of 1's + 2's ( $\leq 11$  in col 1 and 2)  $\leq \lambda_1^{*+1} + \lambda_2^{*+1}$

Continue until we get last diagonal  $\lambda_1^{*+1} + \lambda_2^{*+1} + \lambda_3^{*+1}$

$\lambda_1 + \lambda_2 + \lambda_3 = \# 1's + 2's + 3's \leq (\lambda_1^{*+1}) + (\lambda_2^{*+1}) + (\lambda_3^{*+1})$

• 5 We do not prove the converse  
but do prove a special case

Def Let  $\lambda$  be a partition of  $n$   
where  $n$  is even.  $\lambda$  is  
called a Threshold partition  
if  $\lambda_i = \lambda_i^* - 1$  for  $i=1, \dots, t_r(\lambda)$

Let  $\lambda$  be a threshold partition

$$\begin{matrix} G_x & \begin{matrix} \bullet & \bullet & \bullet & | \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \end{matrix} \end{matrix}$$

Remove the first row and  
column. This removes  $\lambda_1$  and  
subtracts 1 from all other  $\lambda_i$

$$\begin{matrix} \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{matrix} \quad \text{Call it } \lambda'$$

This is another threshold partition  
diagram

Repeat  $\vdots \vdots \lambda'$

For each  $\circ$  at the bottom of

6

By induction  $X'$  has a graph with degree sequence  $X'$ .

$$\text{Ex } X = (5, 4, 4, 3, 3, 1)$$

$$Y \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 & 1 \\ 5 & 1 & 1 & 1 & 0 & 1 \\ 6 & 1 & 1 & 1 & 1 & 0 \end{array}$$

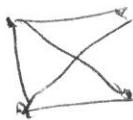
Remove first row  
and column

$$\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}$$

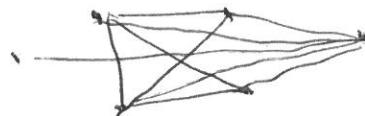
This is threshold  
diagram. We construct  
it's graph



To go back, there  
is an isolated vertex



Now add vertex  
and connect to all



Deg sequence  
(5, 4, 4, 3, 3, 1)

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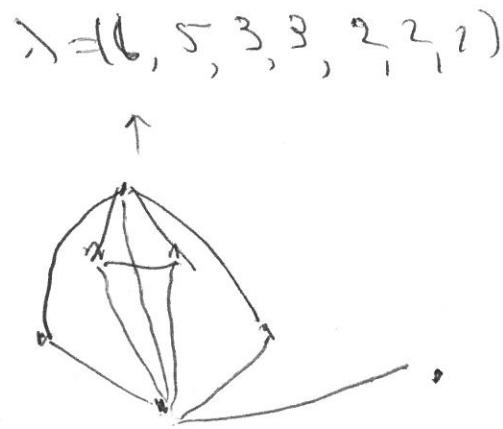
We can also construct the graph not using induction, but reduce graph to trace = 1

$$\lambda = (6, 5, 3, 3, 2, 2, 1)$$

~~9 9 9 9 9 9  
9 9 9 9 9  
9 9 9  
9 9 9 Remove  
row+col  
9 9  
9 9  
9 9  
9 9~~

~~1 1 1 1  
1 1 1 Remove  
row+col  
1 1  
1 1  
1 1  
1 1~~

Add vertex connect to all  
One isolated  
2 isolated vertices!



Add vertex, connect to all  
→ Go to top diagram



8 another example of constructing  
a threshold graph

Ex Diagram : ; ; ;  
; ; ; ;  
; ; ; ;  
 $d = (4, 2, 2, 1, 1)$

$$\lambda_1 = 4 \quad \lambda_2 = 2 \quad \text{trace} = 2$$
$$\lambda_1^* = 5 \quad \lambda_2^* = 3$$

This is a threshold diagram

start with :  $\rightarrow$  —

2 removed dots at bottom of  
first column  $\rightarrow$

— —

..

Now add another vertex and  
connect it to all previous ones

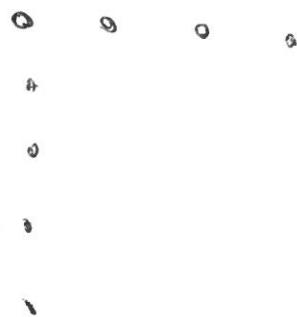


$$d = (4, 2, 2, 1, 1)$$

We will count the number of threshold graphs with  $n$  vertices.

Def An  $n$ -hook is the diagram for  $d = (n-1, 1, \dots, 1)$  where there are  $n-1$  1's

$$\text{Ex } d = (4, 1, 1, 1, 0) \quad n=5$$



Theorem. The number of threshold graphs with  $n$  vertices is  $2^{n-2}$ .  
Let  $n(n)$  be the number of threshold graphs on  $n$  vertices.

$$n(1) = 0 \quad \text{diagram} \quad \text{graph}$$

$$n(2) = 1 \quad \vdots \quad \rightarrow$$

$$n(3) = 2 \quad \vdots \quad \vdots \quad \vdots \quad \begin{array}{c} \nwarrow \\ \nearrow \end{array} \quad \begin{array}{c} \nearrow \\ \nwarrow \end{array}$$

For any  $n$ , a threshold diagram has for its first column and row an  $n$ -hook from the definition of a threshold diagram.

$$\text{Ex } n = 5$$



Removing the  $n$  hook leaves us with an  $n_1$ -hook in the remaining new first row and first column. This continues up to the trace.



To set all threshold graphs with  $n$  vertices (in threshold diagrams) we take all threshold diagrams with less than  $n$  rows and add an  $n$ -hook to each.

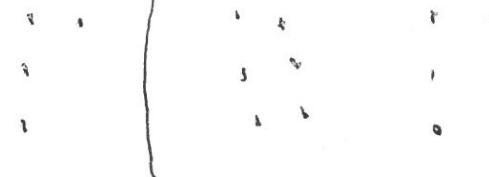
"Threshold Graphs for  $n=5$

$n=1$  empty graph

$n=2$

hook

$n=3$



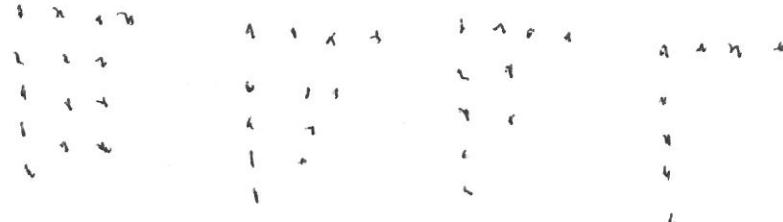
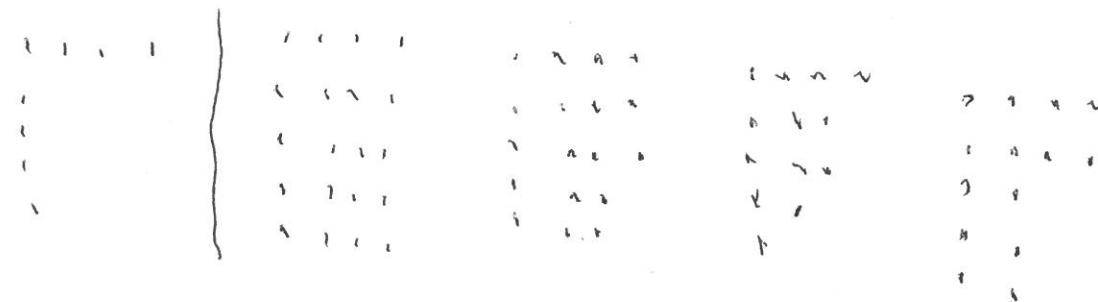
hook

$n=4$



hook

$n=5$



$$2^{5-2} = 2^3 = 8$$