

M A 416

Lesson 33

1. Eulerian Trails

Let $G = (V, E)$ be a graph. Recall that a walk in G is a trail if it has no repeated edges. A trail that contains all the edges in G , each exactly once, is called a Eulerian Trail, due to a famous investigation by Euler (See page 6).

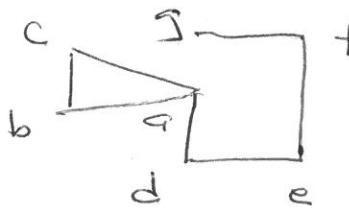
Ex. $G = (V, E)$ $V = \{a, b, c, d, e, f\}$

$E = \{(a, b), (b, c), (c, a), (a, d), (d, e), (e, f), (f, g)\}$

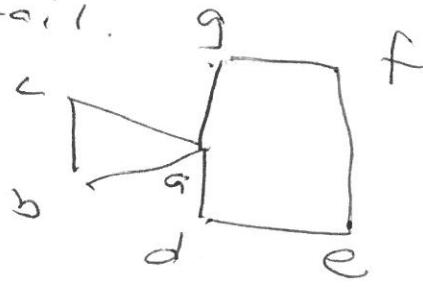
$a - b - c - a$ is a closed trail.

$a - b - c - a - d - e - f - g$ is an open Eulerian Trail.

If we add edge (g, a) to the graph, then $a - b - c - a - d - e - f - g - a$ is a closed Eulerian Trail.

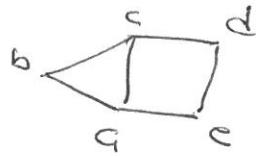


Open



Closed

Notice that in a closed Eulerian trail, the degree of each vertex is even.



Does not have a closed Eulerian Trail.

However it does have a trail that includes all edges, but it is not closed

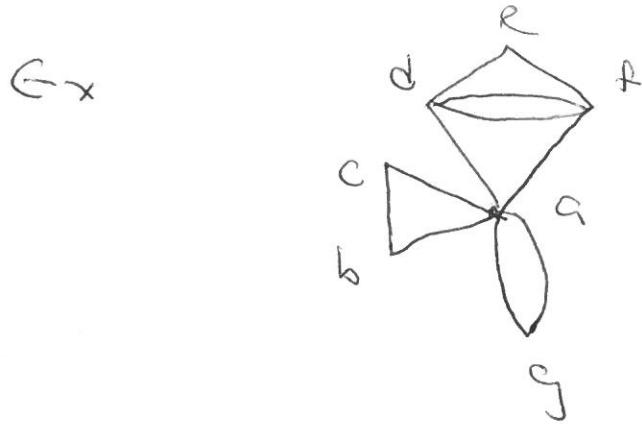
a - b - c - a - e - d - c

Also, the degree(c) = 3 = degree(a)

Theorem A connected graph has a closed Eulerian trail if and only if the degree of each vertex in the graph is even.

Proof Suppose that G has a closed Eulerian Trail Start at a vertex and there is an edge to another vertex. One edge between them says that have degree ≥ 1 . The second vertex has an edge to another vertex and we add 1 to the degrees of each of these vertices. The beginning vertex has 2 edges so far and the end has 1. Continue this, each time leaving a vertex the degree so far is even and arriving at a vertex the degree so far is odd. This stops only when we get back to the last vertex and then all degrees are even.

Conversely suppose all degrees are even. Going from a vertex to the next gives ~~degree~~ one ^{degree} so far. Leaving makes it even and arriving makes the degree odd. The only stopping is when we get back to the original and that is a closed Eulerian Trail.



Start at b

$b - c - a - g - a - d - f - e - \overset{a-b}{d} - f - b$

Is a closed Eulerian Trail. All
vertices have even degree

Theorem. A connected graph G has an open Eulerian Trail if and only if there are exactly two vertices of odd degree

Proof Suppose G has an open Eulerian Trail. Connect the two odd vertices. Now the trail is closed ~~and~~ and all vertices have even degree. Then, the begin and end vertices in the original graph have odd degree.

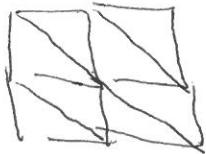
Suppose the degree condition holds. Add an edge between the two vertices of odd degree. Now all edges have

even degree and there is a closed Eulerian Trail. Remove the new edge and we have an open Eulerian Trail

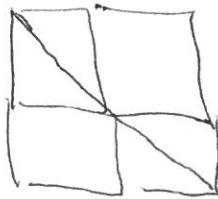
Problem Given a graph, is it possible to trace the edges with a pencil, setting all edges exactly once.

This would require the graph having a closed or open Eulerian Trail.

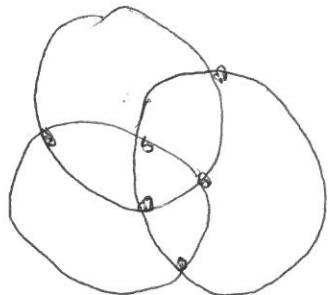
Ex



2 vertices have odd degree, can trace the edges. Open Eulerian Trail



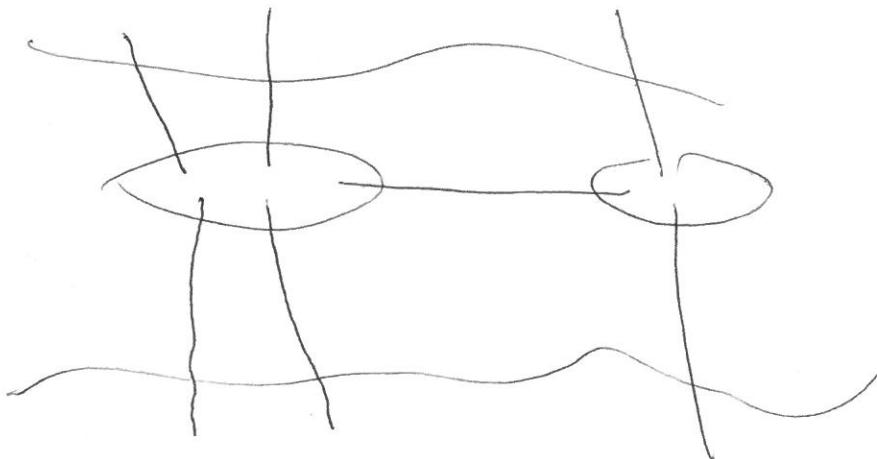
4 vertices have odd degree, not possible



All vertices have even degree. There is a closed Eulerian Trail

Königsberg Bridge Problem

Given the following bridges between land areas



Is it possible to have a walk that crosses all bridges once and ends up on the same land area?

The land areas are the vertices and the bridges are the edges

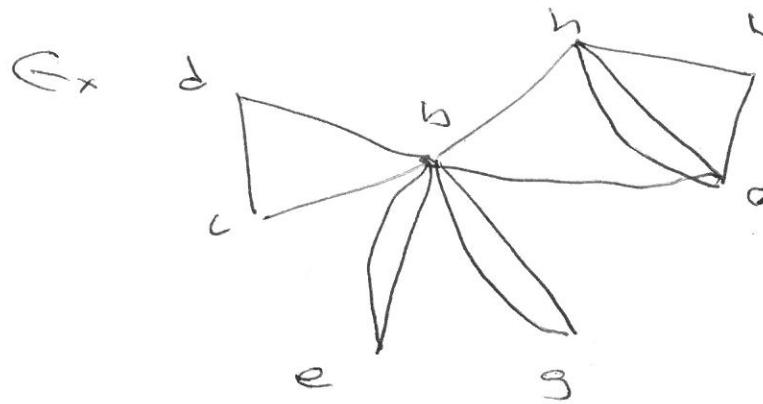


No

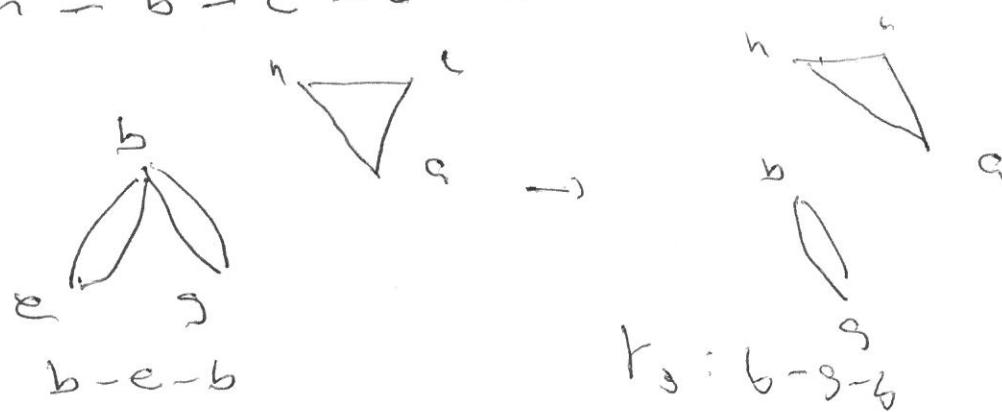
7 Computing Eulerian Trails:

If all edges have even degree in a connected graph, we can obtain a Eulerian trail as follows.

Pick $v_0 \in G$ and compute trail until returning to v_0 . Remove all used edges and call the new edge set E_1 . Pick a vertex in the computed trail that has another edge and repeat the process. Continue this to get all edges. Then insert the new edges trail in the old trail at the vertex where it started.



$$\gamma_1: a - b - c - d - b - h - a$$



$$\begin{array}{c} b \\ \diagdown \\ \Delta \\ \diagup \\ a \end{array} \quad r_1 : b - c - s - h$$

Inserting for one Trail

$$r_1, r_2 \quad a - \underbrace{b - e - b}_{r_2} - c - d - b - h - a$$

$$r_1, r_2, r_3 \quad a - b - \underbrace{g - b}_{r_3} - e - b - c - d - b - h - a$$

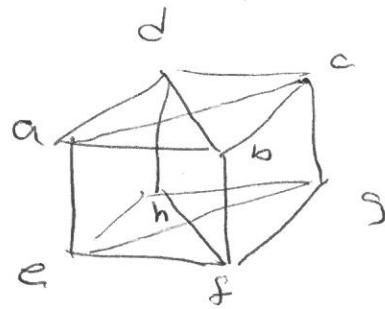
$$r_1, r_2, r_3, r_4$$

$$e - b - g - b - e - b - c - d - b - h - c - a - h - a$$

9

Problems

1.



G is the graph with vertices
the vertices of the cube and
edges as shown.

Can we construct a closed
Eulerian Trail?

If so, find one.

2. Does K_4 have a Eulerian Trail?

Does K_5 have a Eulerian Trail?

Does K_6 ?

Open or Closed

Chapter 11: 1-5, 10-15