

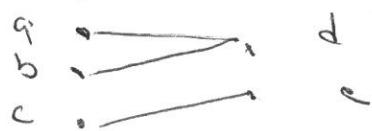
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Lesson 35

Def A graph $G = (V, E)$ is bipartite if V can be partitioned into sets X and Y such that every edge in E has one vertex in X and one vertex in Y

$$\text{Ex } V = \{a, b, c, d, e\} \quad E = \{(a, d), (b, d), (c, e)\}$$

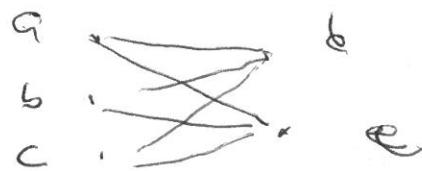
$$X = \{a, b, c\} \quad Y = \{d, e\}$$



The same definition applies to a multigraph.

A bipartite graph is called complete if every vertex in X is adjacent to every vertex in Y

$$\text{Ex } X = \{a, b, c\} \quad Y = \{d, e\}$$



If $|X| = m$ and $|Y| = n$, the complete graph is denoted by $K_{m,n}$.

Theorem. A multigraph is bipartite if and only if all its cycles have even length. \checkmark all cycles have even length

Proof. Suppose G is connected and let $v \in V$. Let $X = \{x \in V, \text{ shortest distance from } v \text{ to } x \text{ is even and } y = \{y \in V, \text{ shortest distance from } v \text{ to } y \text{ is odd}\}$. Clearly $X \cap Y = \emptyset$.

Let s and t be in X and have an edge between them. Let $\omega: v \rightarrow v_2 \dots \rightarrow x$ be the shortest path from v to x and $v - w_2 \dots - y$ be the shortest path from v to y . Then $v - w_2 \dots - x - y - \dots - w_2 - v$ is a cycle of even + 1 + even = odd length. So there are no edges between elements in X . A similar argument shows that there are no edges between elements in Y . Hence G is bipartite.

If G is not connected, apply this argument to each connected component to get the same result.

Suppose that G is bipartite.

Then any path must alternate between X and Y and so a cycle, which ends where it starts, must have an even number of edges.

Ex. Consider an $n \times n$ chessboard where squares are alternately black and white. The squares are the vertices of a graph and the edges are pairs of squares that are adjacent. If X = black squares and Y = white squares, then the graph is bipartite. Can we find a Hamilton path from the bottom left corner to the top right corner?

$$n=2$$

W	B
B	W

To get all vertices requires 3 edges. To go from B to B requires an even number of edges. Not possible.

For n , getting all vertices require $n^2 - 1$ vertical edges; to begin and end with B requires an even number of edges, so it is not possible if n is even.

x

If n is odd, we can take a path that starts in the bottom left, goes across the bottom row, moves one vertically, goes across the second row, moves one vertically and repeats until we get to the top right position

7	8	9
6	5	4
1	2	3

The numbers are the positions of the vertices in the path

1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9

Theorem. Let G be a bipartite graph

with partition sets X and Y

a. If $|X|=|Y|$, Then G does not

have a Hamilton path that begins
and ends in the same set

(To get all vertices, Then begin and
end in different sets)

b. If $|X| \neq |Y|$, Then G does not
have a Hamilton cycle

(The path would have to start in the
larger set and end in the larger
set at a different vertex)

c. If $|X|$ and $|Y|$ differ by 2 or
more, Then G does not have a
Hamilton path

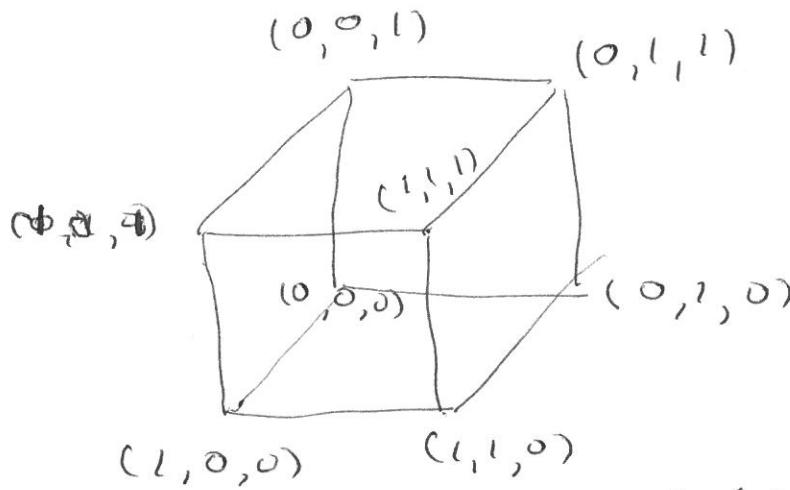
(Such a path would only use
one more vertex in the larger set)

d. If $|X|=|Y|+1$, the G does not
have a Hamilton path that begins
in one and ends in the other

(To have a Hamilton path, we
need to start in X and end in X .)

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Ex φ_3 is the graph whose vertices and edges are those of the 3 dimensional cube



$$\text{Let } X = \{(0,0,0), (1,0,0), (1,0,1), (0,1,1)\}$$

$$Y = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$$

shows that φ_3 is bipartite

A Hamilton cycle

$$(0,0,0) - (1,0,0) - (1,1,0) - (0,1,0) -$$

$$(0,1,1) - (1,1,1) - (1,0,1) - (0,0,1) - (0,0,0)$$

is a Hamilton cycle

More generally, a reflected Gray code, Q_n , is bipartite with X having an even number of ones and Y having an odd number of ones. The method of ~~reflected~~ generating

reflected Gray codes gives a Hamilton cycle (See Sec. 4.3)

Ex $n=4$

0 0 0 0	1 1 0 0
0 0 0 1	1 1 0 1
0 0 1 1	1 1 1 1
0 0 1 0	1 1 1 0
0 1 1 0	1 0 1 0
0 1 1 1	1 0 1 1
0 1 0 1	0 0 1 1
0 1 0 0	1 0 0 1
	1 0 0 0

The edges are between those vertices (4-tuples) that differ in one position

Homework: Ch. 11 29, 30, 39,
47, 48, 49