

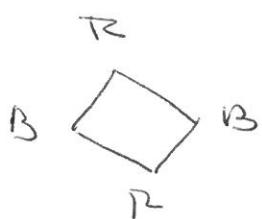
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Lesson 3~~¶~~

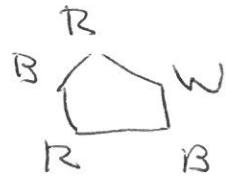
Chromatic Numbers

1 Chromatic Numbers

Def. Let $G = (V, E)$ be a graph and S be a set of elements called colors. A coloring of G is an assignment of colors to each vertex in G such that adjacent vertices are assigned different colors. The assignment is called a k -coloring when k elements from S are used.



2-coloring



3-coloring



4-coloring

For a given G what is the smallest number of colors that can be used to color G ? This number is denoted by $\chi(G)$.

In the above 3 graphs,
 $\chi(G) = 2, 3$ and 4 respectively

2 We have If G is

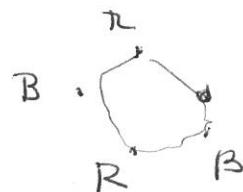
- a A null graph, $\chi(G) = 1$
- b A complete graph, K_n , $\chi(G) = n$
- c. An even cycle, $\chi(G) = 2$
- d An odd cycle $\chi(G) = 3$
- e A tree $\chi(G) = 2$

Proof. a. There are no edges so each vertex can get the same color

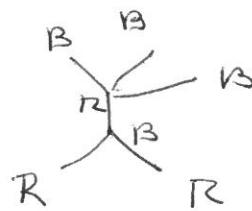
b. Each vertex is connected adjacent to every other vertex so $\chi(G) = n$

c. Alternates 2 colors, $\chi(G) = 2$

d. Alternates 2 colors but the last assignment has to have a third color:



e. Alternates colors as one moves along the tree



Problem. Color

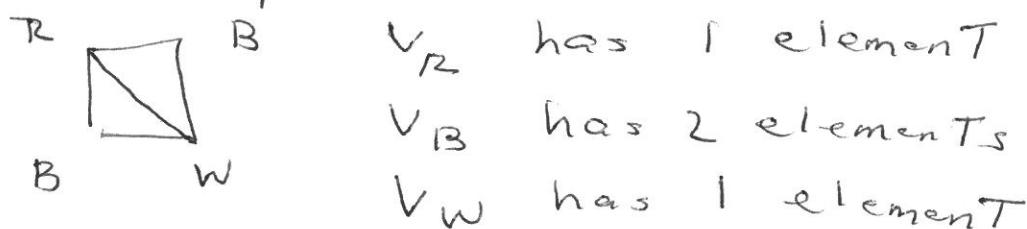


3

Let G be a graph and $S = \{1, \dots, k\}$ be used to color G . Define
 $V_i = \{v \in V; v \text{ is colored } i\}$

Each V_i is a null graph

The collection V_1, \dots, V_k is called
a color partition of G .



Theorem. Let $G = (V, E)$, $|G| = n$.

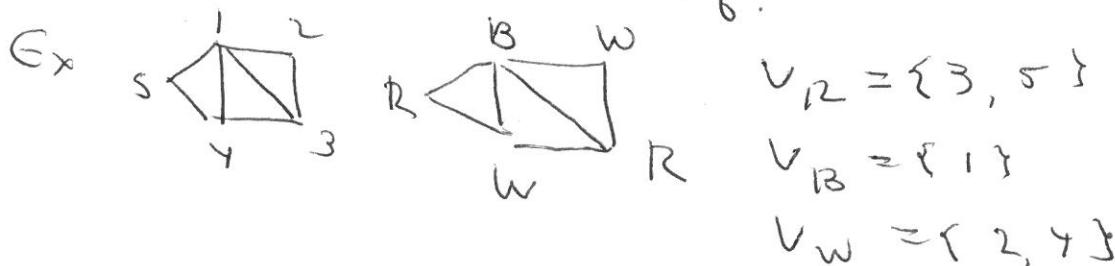
Let $\chi(G) = k$ with color partition

V_1, \dots, V_k . Suppose $|V_1| \geq |V_2| \geq \dots \geq |V_k|$

Let $g = |V_1|$. Then $\chi(G) \geq n/g$

$$\begin{aligned} \text{Proof. } n &= |V| = |V_1| + \dots + |V_k| \\ &\leq |V_1| + \dots + |V_1| = kg \end{aligned}$$

Then $\chi(G) = k \geq n/g$.



$$n = 5 \quad g = 2 \quad k = 3$$

$$3 = \chi(G) = k \geq \frac{n}{g} = \frac{5}{2}$$

4

Thm. Suppose G has at least one edge. $\chi(G) = 2$ if and only if G is bipartite.

Proof If G is bipartite, G can be partitioned into sets X and Y with no edges between elements in the same set. Hence all elements in X can be colored G and elements in Y can be colored W . $\chi(G)$ is not 1 because there is an edge. Hence $\chi(G) = 2$.

If $\chi(G) = 2$, let $X = \{ \text{all vertices colored } G \}$, $Y = \{ \text{all vertices colored } W \}$. There are no edges between vertices in X and none between vertices in Y , hence G is bipartite.

Thm. Suppose G has at least one edge. Then $\chi(G) = 2$ if and only if every cycle has even length.

Proof Every cycle has even length
 \uparrow
 G is bipartite
 \uparrow
 $\chi(G) = 2$

The next result gives a method for finding an upper bound for $\chi(G)$. In particular, $\chi(G) \leq \Delta + 1$ where Δ is the maximum of the degrees for the vertices of G . (Recall, the degree of a vertex = the number of edges incident with it).

Let $V = \{v_1, \dots, v_n\}$

Let $1, 2, \dots$ be colors available

Assign 1 to v_1 and let $X_1 = \{v_1\}$

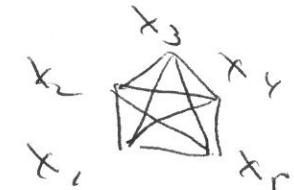
Assign the smallest possible number to x_2 so that we have a coloring for

$X_2 = \{v_1, v_2\}$. This is 1 or 2, depending on the number of edges at v_1 to elements in X_2 . Assign the smallest number possible to v_3 so that $X_3 = \{v_1, v_2, v_3\}$

is colored. That is one more than the max number of edges at any vertex v_1, v_2 . So it is ≤ 3

Assign the smallest number to v_4
 so that $X_4 = \{v_1, v_2, v_3, v_4\}$ is colored
 and it is one more than the
 max number of edges at any of
 v_1, v_2, v_3 . So it is ≤ 4 . Continuing
 to v_n which is assigned the
 number greater than the max number
 of edges for vertices in $X_{n-1} = \{v_1, v_{n-1}\}$
 That is $\Delta + 1$.

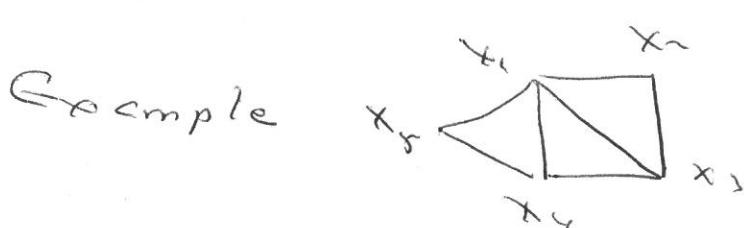
Example $G = K_5$



Vertex	Color
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

$\Delta = 4$

$\chi(G) = 5 \leq \Delta + 1$

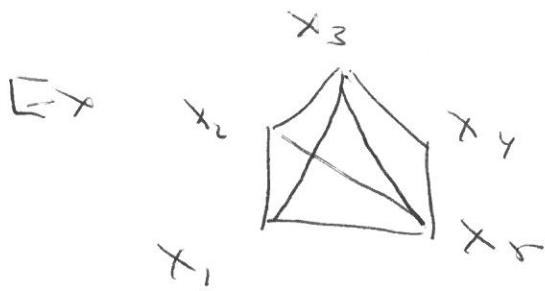


Vertex	Color
x_1	1
x_2	2
x_3	3
x_4	2
x_5	3

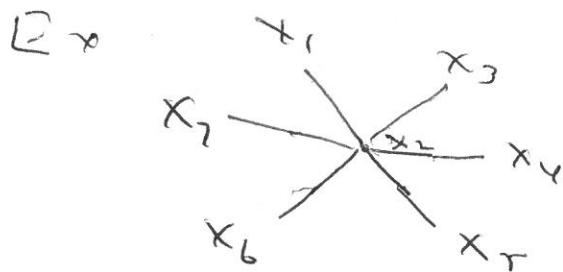
$\Delta = 4$

$\chi(G) = 3 \leq \Delta + 1$

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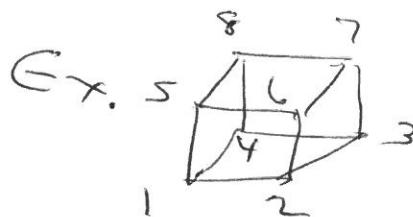


make the table
what is Δ and $\chi(G)$?



make the table
what is Δ and $\chi(G)$?

Notice that $\chi(G)$ can be
a lot less than our bound
of $\Delta + 1$. In fact $\chi(G) = \Delta + 1$ if and only if
 $G = K_n$ or $G = C_n, n, \text{odd}$



make THE TABLE
what is Δ , $\chi(G)$?
Do you see several of our Theorems?