

MA 416

LESSON 38

Chromatic Functions I

Chromatic Functions

$$P_G(k)$$

Recall that a coloring of G with elements from a set S is an assignment of an element from S to each vertex in G such that if two vertices are adjacent (connected by an edge) then the assigned colors are different. The ~~smallest~~ smallest number of colors that can color G is called the chromatic number of G and denoted by $\chi(G)$.

We found $\chi(G)$ for some graphs

Graph	$\chi(G)$
Null graph	1
Tree	2
Complete K_n	$n!$
Even cycle	2
Odd cycle	3

Today we consider a different, and harder, problem. For $G = (V, E)$, $|V|=n$, and S , $|S|=k$ how many k colorings exist?

Call this number $P_G(k)$. Clearly, if $k < \chi(G)$, then $P_G(k) = 0$.

Ex Let G be a tree, $|G|=n$.

Pick any $v \in G$. v can be colored by any of k colors.

Any vertex adjacent to v can be colored by $k-1$ colors. This process repeats so that any of the next vertices can be colored in $k-1$ ways. So $P_G(k) = k(k-1)^{n-1}$ for any tree

Ex If N is the null graph, then there are no restrictions, so $P_N(k) = k^n$ (where $|N|=n$, $|S|=k$)

\Rightarrow Let $G = K_n$. A first vertex can be colored in k ways. A second can be colored in $k-1$ ways. A third vertex, adjacent to the first two, can be colored in $k-2$ ways. Continuing, the last vertex can be colored in $k-(n-1)$ ways since it is adjacent to $n-1$ vertices, all colored differently. Hence

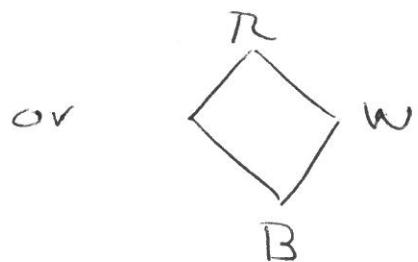
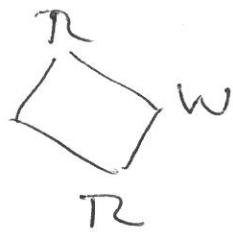
$$P_G(k) = k(k-1)\dots(k-n+1) = \frac{k!}{(k-n)!}$$

Generally, when cycles exist, the problem is harder. Consider C_4

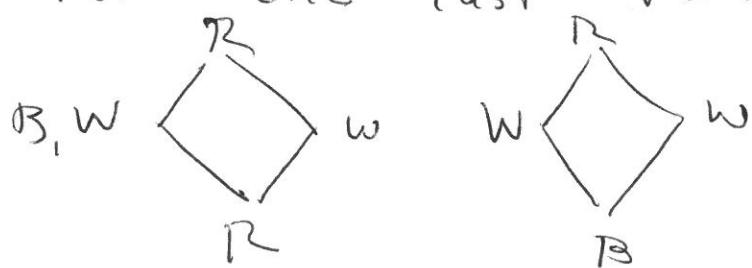
Ex



For example, suppose $S = \{R, W, B\}$



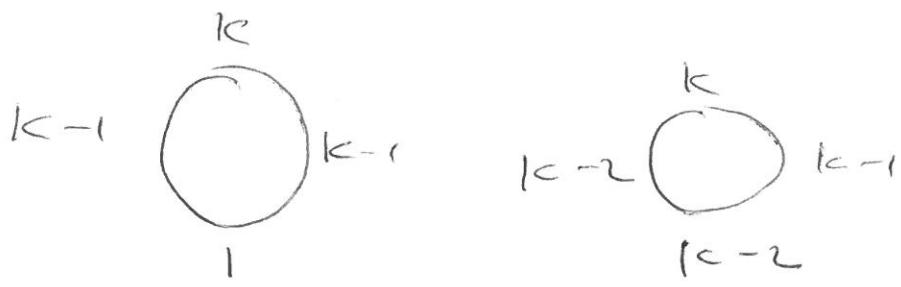
We set different numbers of choices
for the last vertex



So, in general how many choices
there are for the last vertex
depends on what happens at
the third vertex So, our
assigning of numbers to vertices
becomes unclear.

In the first case, there are
3 · 2 · 1 · 2 choices in the second
3 · 2 · 1 · 1 choices

For k colors, the choices
would be



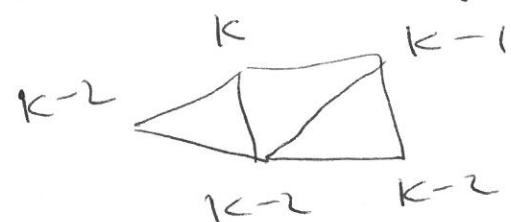
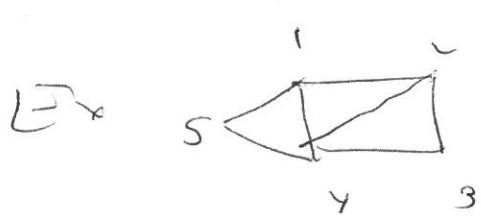
In the first case, the color at the top has been repeated at the bottom, giving $k-1$ choices on the left. In the second, the color at the top is different from the color at the bottom, giving $k-2$ choices on the left.

Hence

$$P_G(k) = k(k-1)^2 + k(k-1)(k-2)^2$$

As you can guess, if this simple example requires cases, things can get complicated

Notice that, so far, $P_G(k)$ is a polynomial. This is always the case. It is called the chromatic polynomial and its values for k give the number of colorings for G using k colors.



We assign colors in the order as shown in the first graph. This is the answer for the number

of colorings

$$P_G(k) = k(k-1)(k-2)^3$$

It would have been less clear if the order of assignment was 1-2-4-3-5 (why?)

Note that

$$P_G(1) = 0$$

$$P_G(2) = 0$$

$$P_G(3) = 3 \cdot 2 \cdot 1 = 6$$

With 3 colors, there are 6 colorings.
With 2 colors, there are 0.

We see next an algorithm to compute $P_G(k)$. Each step has 2 parts, so it can be lengthy.

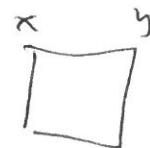
The idea is to reduce G to null graphs, each null graph will have a certain (perhaps different) number of vertices, say k_1, \dots, k_j . Then these can be collected in k^{n_1}, \dots, k^{n_j} ways respectively. The numbers will be added or subtracted to get the final $P_G(k)$. If there are null graphs of order 4 and 5, say, then the monomials will be k^4 and k^5 . Those will be added or subtracted. A miracle is that all null graphs with the same number of vertices will all be added (or subtracted).

$P_G(k)$ will show a lot of facts about G .

Chromatic Polynomial Algorithm

We do this until there are no edges left.

1. Pick $x, y \in V$, $x-y \in E$



2. Remove $x-y$



Call this new graph G_1

3. G_1 has 2 possibilities

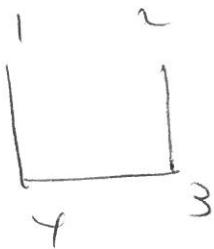
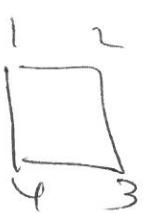
Either x and y are colored differently or the same

(in G_1 , not G)

4. Let $C(k)$ be the graphs in G_1 where
x and y are colored differently

4. Let $C(k)$ be the graphs in G_1
where x and y are colored differently

Then $C(k)$ is in 1-1 correspondence
with coloring in G (since in G
x and y are colored differently)



Assign graphs on right to graphs on left. It is 1-1 and onto

This is done by adding edge 1-2 and is possible since colors at 1 and 2 are different.

5. $D(k)$ = graphs in G_1 where x and y are the same.

Fuse x and y with all else being the same



(colors at 1 and 2 are the same)

$$\text{Now } P_{G_1}(k) = C(k) + D(k)$$

$$= P_G(k) + D(k)$$

$$\rightarrow P_G(k) = P_{G_1}(k) - D(k)$$

We are computing $P_G(k)$. We have reduced it to computing $P_{G_1}(k)$ and $D(k)$. In both of these, the number of edges is

less than the number of edges in
in $P_{G_i}(k)$. So if we repeat this
enough times, we will get the
desired null graphs. Just what
we want. So using the repeated
equation (like $P_{G_i}(k) = P_{G_i}(k) - D(k)$)
 $P_{G_i}(k)$ will be a combination of the
monomials obtained in the process

A very important thing to
notice is that the number
of vertices does not drop in
 $P_{G_i}(k)$ but drops by one in
 $D(k)$. Also in going to $P_{G_i}(k)$
the sign does not change but
does in going to $D(k)$.

So in the process every time we
reduce a vertex the sign changes.
The ramification is that if
the number of vertices in any of
the null graphs has the same

Parity as n , the sign is positive
 but if it has different parity
 from n , the sign is negative
 when constructing $P_G(k)$ from
 the null graphs

Ex Suppose $|G|=7$ and the
 final null graphs are

$$\begin{matrix} \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} & \textcircled{6} & \textcircled{7} \\ \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} & \textcircled{6} & \textcircled{7} \\ \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} & \textcircled{6} & \textcircled{7} \end{matrix}$$

Then

$$P_G(k) = x^7 - x^4 + 3x^3.$$

$$\begin{aligned}
 P_{G_1}(k) &= C + D \\
 &= C + D' \\
 P_G(k) &= P_{G_1}(k) - P(k) \\
 &= k^7 - k^4
 \end{aligned}$$

$\exists x \quad G \rightarrow$

$$G_1 \circ \circ = \cancel{\text{---}} + A \quad B \quad A \quad A$$

$$= G + \underset{A}{\bullet}$$

$$\rightarrow G = G_1 - \underset{A}{\bullet}$$

$$= K^2 - K$$

$\exists x$

$$G_1 = \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} A \\ | \\ B \end{array} + \begin{array}{c} A \\ | \\ A \end{array}$$

$$= \begin{array}{c} \diagup \\ G \end{array} + \begin{array}{c} \diagdown \\ A \\ \text{fused} \end{array}$$

$$\begin{array}{c} \diagup \\ = \diagdown - \text{---} \end{array}$$

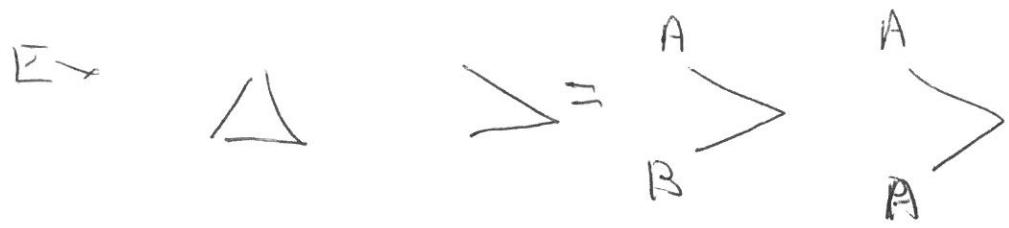
$$K(K^2 - K) = (K^2 - K)$$

The $K(K^2 - K)$ comes from that
 The graph is disconnected with
 components X and Y and
 $P_X(K) = K$ $P_Y(K) = (K^2 - K)$ and
 the product of the component
 polynomials is the polynomial of the graph

Properties of $P_G(k)$

1. $\deg P_G(k) = \text{number of vertices}$
2. coefficient of k^{n-1}
 $-2 = \text{number of edges}$
3. The constant Term is 0
4. The term k appears which say
 G is connected.

The algorithm gets complicated fast. Any computer algebra package would have the algorithm. The algorithm also shows that $P_G(k)$ is a polynomial in k .



$$G = \Delta + \overbrace{\quad}^{\substack{A \\ \text{fused}}}$$

$$\begin{aligned}\Delta &= G - \overbrace{\quad} \\ &= [(k^3 - k^2) - (k^2 - k)] - (k^2 - k) \\ &= k^3 - 3k^2 + 2k\end{aligned}$$

\rightarrow 3 vertices

3 edges

connected