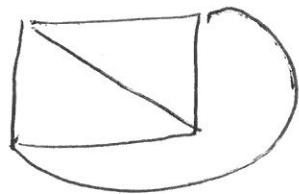


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Lesson 40

A planar graph is a graph which can be drawn in the plane in such a way that pairs of edges only intersect at vertices K_4 is a planar graph



K_n , $n > 4$, is not a planar graph as we will see.

We consider not only vertices and edges of planar graphs, but also regions that are bounded by the edges. In K_4 there are 4 vertices, 6 edges and 4 regions including the infinite region outside the graph

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Let G be a planar graph with n vertices, e edges and r regions. Let f_1, \dots, f_r be the number of edges bounding the regions $1, \dots, r$. Then $f_1 + f_2 + \dots + f_r = 2e$ as each edge borders 2 regions.

Theorem (Euler) Let G be a plane graph of order n ($n = n$), e edges and r regions. Then

$$r - e + n = 2.$$

Proof. If G is a tree, $e = n - 1$ and $r = 1$. Hence the result holds for trees. For general G , let T be a spanning tree for G . So for T , $r' - e' + n' = 2$. We move from T to G by putting back one edge at a time. Each time we do it, e increases by 1, r increases by 1 and e stays the same.

3 Hence the equation holds every time we put back an edge, all the way up to G. So the result holds

Theorem. Let G be a connected planar graph. The same vertex in G has at most 5 incident edges (The degree is at most 5)

Proof. Any region is bounded by at least 3 edges, hence $f_i \geq 3$ for each $i \in S$.

$$2e = f_1 + \dots + f_r \geq 3r + 3 = 3r$$

$$\rightarrow \frac{2e}{3} \geq r$$

By Euler's formula

$$\frac{2e}{3} \geq r = e - n + 2$$

$$\rightarrow e \leq 3n - 6$$

If d_1, \dots, d_n are the degrees of v_1, \dots, v_n , then $d_1 + \dots + d_n = 2e$

$$\rightarrow \frac{d_1 + \dots + d_n}{n} = \frac{2e}{n} \leq \frac{6n - 12}{n} < 6$$

4. Since the average of the degrees is less than 6,
at least one of them is
less than or equal to 5

Ex K_n is planar if $n \leq 4$

Proof. We saw that K_4 is planar

Consider K_5 . By the last result

$$e \leq 3n - 6$$

For K_5 , $n=5$, $e=10$, so

the inequality does not hold

and K_5 is not planar. Hence

$\forall K_n, n \geq 5$ is not planar since
it contains copies of K_5
as subgraphs

Recall that a bipartite graph

is one that has 2 components

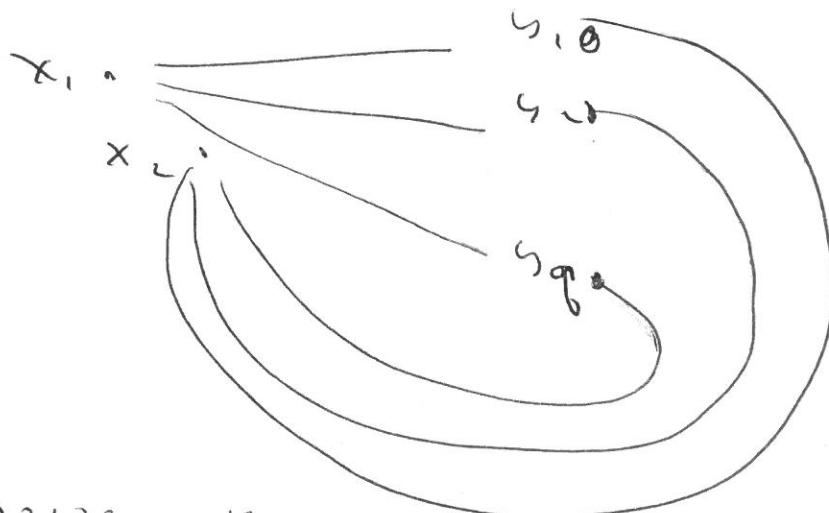
X and Y such that X and Y
are null graphs. Suppose $|X|=p$,
 $|Y|=q$. If between each $x \in X$ and
 $y \in Y$, there is an edge, G is
called a complete bipartite graph
and is denoted by $K_{p,q}$.

Clearly, G has $p \cdot q$ edges

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$\Leftarrow K_{p,q}$ is planar if $p \leq 2$ or $q \leq 2$.

If $p = 2$



Shows that G is planar

Consider $K_{3,3}$. G has no cycles of length 3 since G is bipartite. Hence each region is bounded by at least 4 edges.

Hence each $f_i \geq 4 \Rightarrow$

$$2e = f_1 + f_r \geq 4 + \dots + 4 = 4r$$

$$\rightarrow r \leq \frac{e}{2} \text{ . Then }$$

$$\frac{e}{2} \geq r = e - n + 2 \text{ and}$$

$$2n - 4 \geq e$$

$K_{3,3}$ has $n=6$ vertices and $e=9$

edges. This contradicts the inequality

Therefore $K_{3,3}$ is not planar and
so $K_{p,q}$, $p, q \geq 3$ is not planar
either since it has a copy of
 $K_{3,3}$ in it

Theorem. The chromatic number of a planar graph is at most 6

Proof Suppose there is a planar graph whose chromatic number is at most 7 or more. G has a vertex v whose degree is at most 5. Remove v and all incident edges to get G' . G' has order one less than G . By induction G' has a coloring. Putting v back in with its edges, there are at most 5 colors ~~not~~ used in vertices adjacent to v so v can have the 6th color and G is 6 colored.

Lemma. Suppose there is a k coloring of graph H . Suppose red and blue are 2 of them. Let $W =$ all vertices colored either red or blue and $H_{r,b}$ be the subgraph induced by the vertices in W . Let $C_{r,b}$ be a connected component of $H_{r,b}$. Interchange colors red and blue assigned to the vertices of $C_{r,b}$ and we get another k coloring of H .

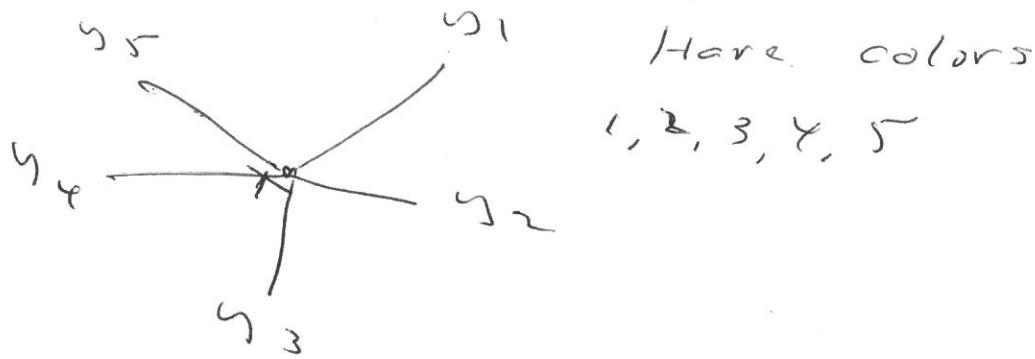
Proof After the change of colors, if both adjacent vertices are inside $C_{r,b}$ they still have different colors (just reversed). If one vertex is outside $C_{r,b}$ and the other is not, they are not connected so we still have a coloring. If both are outside $C_{r,b}$, the colors do not change and we are O.K.

Note: We are showing that after the change in $C_{r,b}$, that no edges in H are adjacent to vertices of the same color. So we have a coloring still.

Theorem. The chromatic number of a planar graph is at most 5.

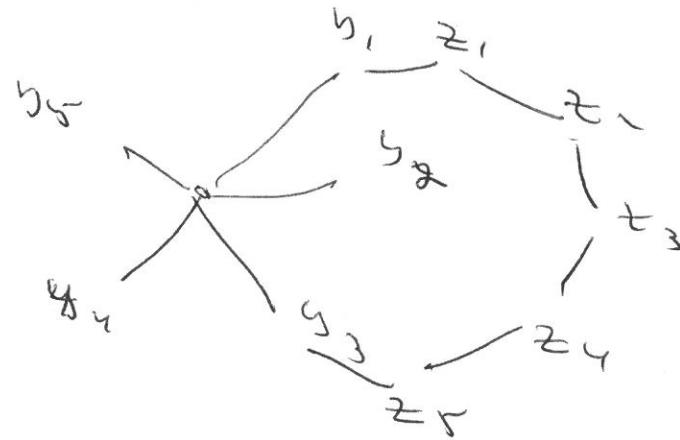
Proof. Let G be a planar graph, $|G|=n$. If $n \leq 5$, then the result is clear. Suppose $n > 5$. Induct on n . There is a vertex $x \in G$, $\deg(x) \leq 5$. If $\deg(x) < 5$, then there is a color not assigned to a vertex adjacent to x so use it and we are done. So $\deg(x) = 5$. Let H be the subgraph obtained by removing x and all incident edges. By induction there is a 5-coloring of H . There are 5 vertices adjacent to x . If some have the same color, this frees up a color for x and we are done. So all 5 vertices adjacent to x have different colors.

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Consider subgraph H_{13} of all vertices of H assigned 1 or 3 and all incident edges. If y_1 and y_3 are in different connected components, change colors in the component with y_1 and still get a coloring. Now only 4 colors are used in the 5 vertices and color x the fifth. Assume y_1 and y_3 are in the same connected component. There is a path $\gamma: y_1 - z_1 - \dots - z_k - y_3$, alternating colors. Then $x - y_1 - z_1 - \dots - z_k - y_3 - x$ determines a closed curve Γ . y_2 and y_4 are on opposite sides of Γ . Let $H_{2,4}$ be graph induced by colors 2 and 4. Then y_2 and y_4 are not in the same connected

12 component of $H_{2,y}$ by the relation
 to the closed curve with y_1 and y_3
 so switching colors in the component
 containing y_2 will still be a
 coloring and y_2 and y_4 will have
 the same color. Again, and for the
 final time, this allows us to use
 the fifth fifth color for x .



Overview of Proof

The idea in this proof is to use induction. There is a graph vertex of degree 5. Remove x and the ~~order~~^X of what remains is one less so it can be 5-colored. If each vertex adjacent to x have some color repeated, we are done because x can get the left over color. Let 2 of the colors be R and B and take the subgraph of vertices colored R and B and then take the connected component of it with these vertices. There is a lemma which shows interchanging R and B in the graph is still a coloring. If R and B are in the 2 of the vertices are in different components, change the colors in one and the adjacent vertices will only have 4 colors so x can get the 5th. If not, then some of the other 3 of the 5 vertices are not connected (due to planar graphs) and colors can be changed in them and a color again is freed up.